

The symmetric determinant for $n \times n$ matrices and the symmetric Newton formula in the 3×3 case

J. Szigeti and L. van Wyk

ABSTRACT. One of the aims of this paper is to provide a short survey on the natural left, right and symmetric generalizations of the classical determinant theory for square matrices with entries in an arbitrary (possibly non-commutative) ring. This will put us in a position to give a motivation for our main results. We use the preadjoint matrix to exhibit a general trace expression for the symmetric determinant. The symmetric version of the classical Newton trace formula is also presented in the 3×3 case.

1. INTRODUCTION

Let S_n denote the symmetric group of all permutations of the set $\{1, 2, \dots, n\}$. For an $n \times n$ matrix $A = [a_{i,j}]$ over an arbitrary (possibly non-commutative) ring or algebra R with 1, the element

$$\begin{aligned} \text{sdet}(A) &= \sum_{\tau, \rho \in S_n} \text{sgn}(\rho) a_{\tau(1), \rho(\tau(1))} \cdots a_{\tau(t), \rho(\tau(t))} \cdots a_{\tau(n), \rho(\tau(n))} \\ &= \sum_{\alpha, \beta \in S_n} \text{sgn}(\alpha) \text{sgn}(\beta) a_{\alpha(1), \beta(1)} \cdots a_{\alpha(t), \beta(t)} \cdots a_{\alpha(n), \beta(n)} \end{aligned}$$

of R can be obviously considered as the symmetric determinant of A .

One of the aims of this paper is to provide a short survey on the natural left, right and symmetric generalizations of the classical determinant theory. The starting point is the above definition of $\text{sdet}(A)$. We collect and explain some known results and point out some similarities and differences between the (traditional) commutative base ring case and the general case. This will put us in a position to give a motivation for our main results. First we prove that $\text{sdet}(A) = \text{tr}(AA^*) = \text{tr}(A^*A)$, where $\text{tr}(A)$ is the sum of the diagonal entries and A^* is the so called

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preadjoint matrix of the $n \times n$ matrix $A \in M_n(R)$. Then we present the following symmetric version of the Newton trace formula for a 3×3 matrix $A \in M_3(R)$:

$$\text{sdet}(A) = \text{tr}^3(A) - \text{tr}(A) \cdot \text{tr}(A^2) - \text{tr}(A \cdot \text{tr}(A) \cdot A) - \text{tr}(A^2) \cdot \text{tr}(A) + \text{tr}(A^3) + \text{tr}((A^\top)^3),$$

where A^\top denotes the transpose of A . The symmetric characteristic polynomial of this A and the corresponding general Cayley-Hamilton identity are also presented by traces.

2. PRELIMINARIES

The preadjoint matrix $A^* = [a_{r,s}^*]$ of an $n \times n$ matrix $A = [a_{i,j}]$ (over an arbitrary ring or algebra R with 1) is defined as the following natural symmetrization of the classical adjoint:

$$\begin{aligned} a_{r,s}^* &= \sum_{\tau, \rho} \text{sgn}(\rho) a_{\tau(1), \rho(\tau(1))} \cdots a_{\tau(s-1), \rho(\tau(s-1))} a_{\tau(s+1), \rho(\tau(s+1))} \cdots a_{\tau(n), \rho(\tau(n))} \\ &= \sum_{\alpha, \beta} \text{sgn}(\alpha) \text{sgn}(\beta) a_{\alpha(1), \beta(1)} \cdots a_{\alpha(s-1), \beta(s-1)} a_{\alpha(s+1), \beta(s+1)} \cdots a_{\alpha(n), \beta(n)}, \end{aligned}$$

where the first sum is taken over all $\tau, \rho \in S_n$ with $\tau(s) = s$ and $\rho(s) = r$ (the second sum is taken over all $\alpha, \beta \in S_n$ with $\alpha(s) = s$ and $\beta(s) = r$). We note that the (r, s) entry of A^* is exactly the signed symmetric determinant $(-1)^{r+s} \text{sdet}(A_{s,r})$ of the $(n-1) \times (n-1)$ minor $A_{s,r}$ of A arising from the deletion of the s -th row and the r -th column of A . If R is commutative, then $A^* = (n-1)! \text{adj}(A)$, where $\text{adj}(A)$ denotes the ordinary adjoint of A .

The right adjoint sequence $(P_k)_{k \geq 1}$ of A is defined by the recursion: $P_1 = A^*$ and $P_{k+1} = (AP_1 \cdots P_k)^*$ for $k \geq 1$. Originally the k -th right determinant was defined as the top left entry of the product matrix $AP_1 \cdots P_k$. These definitions were introduced in [8].

The above mentioned k -th right determinant is not invariant with respect to the conjugate action of $\text{GL}_n(Z(R))$ on $M_n(R)$ (here $Z(R)$ denotes the centre of R). A more appropriate (and invariant) definition for the k -th right determinant is the trace of $AP_1 \cdots P_k$ (see [1] and [10]):

$$\text{rdet}_{(k)}(A) = \text{tr}(AP_1 \cdots P_k).$$

The left adjoint sequence $(Q_k)_{k \geq 1}$ can be defined analogously: $Q_1 = A^*$ and $Q_{k+1} = (Q_k \cdots Q_1 A)^*$ for $k \geq 1$. The k -th left determinant of A is

$$\text{ldet}_{(k)}(A) = \text{tr}(Q_k \cdots Q_1 A).$$

Note that $\text{rdet}_{(k+1)}(A) = \text{rdet}_{(k)}(AA^*)$ and $\text{ldet}_{(k+1)}(A) = \text{ldet}_{(k)}(A^*A)$. The basic properties of these determinants are given in the following theorems.

Theorem 2.1. (see [1],[10]) *If $T \in \text{GL}_n(Z(R))$ is an invertible matrix with entries in the centre $Z(R)$ of R , then*

$$\text{tr}(T^{-1}AT) = \text{tr}(A), \quad (T^{-1}AT)^* = T^{-1}A^*T,$$

$$\text{rdet}_{(k)}(T^{-1}AT) = \text{rdet}_{(k)}(A), \quad \text{ldet}_{(k)}(T^{-1}AT) = \text{ldet}_{(k)}(A).$$

The next results shed light on the fact that we call $\text{radj}_{(k)}(A) = nP_1 \cdots P_k$ the k -th right adjoint and $\text{ladj}_{(k)}(A) = nQ_k \cdots Q_1$ the k -th left adjoint of A .

Theorem 2.2. (see [8],[10]) *The product matrices $\text{Aradj}_{(1)}(A)$ and $\text{ladj}_{(1)}(A)A$ in $M_n(R)$ can be written as*

$$\text{Aradj}_{(1)}(A) = nAA^* = \text{tr}(AA^*)I + C' = \text{rdet}_{(1)}(A)I + C'$$

and

$$\text{ladj}_{(1)}(A)A = nA^*A = \text{tr}(A^*A)I + C'' = \text{ldet}_{(1)}(A)I + C'' ,$$

where I is the identity matrix, $\text{tr}(C') = \text{tr}(C'') = 0$ and all entries of the matrices C' and C'' are in the additive commutator subgroup $[R, R]$ of R generated by all elements of the form $[u, v] = uv - vu$, $u, v \in R$.

Theorem 2.3 (see [8]) *If the ring R satisfies the polynomial identity*

$$[[[\dots [x_1, x_2], x_3], \dots], x_k], x_{k+1}] = 0$$

(R is Lie nilpotent of index k), then the products $\text{Aradj}_{(k)}(A)$ and $\text{ladj}_{(k)}(A)A$ are scalar matrices in $M_n(R)$ such that

$$\text{Aradj}_{(k)}(A) = nAP_1 \cdots P_k = \text{rdet}_{(k)}(A)I, \text{ladj}_{(k)}(A)A = nQ_k \cdots Q_1A = \text{ldet}_{(k)}(A)I.$$

If R is commutative, then $\text{radj}_{(1)}(A) = \text{ladj}_{(1)}(A) = nA^* = n!\text{adj}(A)$ and

$$\text{rdet}_{(k)}(A) = \text{ldet}_{(k)}(A) = n \{(n-1)!\}^{1+n+n^2+\dots+n^{k-1}} \{\det(A)\}^{n^{k-1}}.$$

Let $1 \leq t \leq n-1$ be an integer and $R = R_0 \oplus R_1$ be a \mathbb{Z}_2 -grading of R ($R_i R_j \subseteq R_{i+j}$, $i, j \in \mathbb{Z}_2$). Now $A \in M_n(R)$ is called an (n, t) supermatrix if

$$a_{i,j} \in R_0 \text{ for all } 1 \leq i, j \leq t \text{ and } t+1 \leq i, j \leq n,$$

and

$$a_{i,j} \in R_1 \text{ for all } 1 \leq i \leq t, t+1 \leq j \leq n \text{ and } t+1 \leq i \leq n, 1 \leq j \leq t.$$

Thus an (n, t) supermatrix can be partitioned into square and rectangular blocks as follows:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ is a $t \times t$ and $A_{2,2}$ is an $(n-t) \times (n-t)$ square matrix over R_0 and $A_{1,2}$ is a $t \times (n-t)$ and $A_{2,1}$ is an $(n-t) \times t$ rectangular matrix over R_1 . Clearly, the set of all (n, t) supermatrices $M_{n,t}(R)$ is a subring (algebra) of $M_n(R)$.

Theorem 2.4. (see [9]) *If $R = R_0 \oplus R_1$ is a \mathbb{Z}_2 -grading of R and $A \in M_{n,t}(R)$, then $A^* \in M_{n,t}(R)$ and $\text{rdet}_{(k)}(A), \text{ldet}_{(k)}(A) \in R_0$ for all $1 \leq k$.*

Let $R[z]$ denote the ring of polynomials of the single commuting indeterminate z , with coefficients in R . The k -th right (left) characteristic polynomial of A is the k -th right (left) determinant of the $n \times n$ matrix $zI - A$ in $M_n(R[z])$:

$$p_{A,k}(z) = \text{rdet}_{(k)}(zI - A) \text{ and } q_{A,k}(z) = \text{ldet}_{(k)}(zI - A).$$

Theorem 2.5. (see [9]) *If $R = R_0 \oplus R_1$ is a \mathbb{Z}_2 -grading of R and $A \in M_{n,t}(R)$, then $p_{A,k}(z), q_{A,k}(z) \in R_0[z]$ for all $1 \leq k$.*

The above characteristic polynomials appear in the following Cayley-Hamilton theorems.

Theorem 2.6. (see [10]) *The first right characteristic polynomial $p_{A,1}(z) \in R[z]$ of a matrix $A \in M_n(R)$ is of the form*

$$p_{A,1}(z) = \lambda_0^{(1)} + \lambda_1^{(1)}z + \cdots + \lambda_{n-1}^{(1)}z^{n-1} + \lambda_n^{(1)}z^n$$

with $\lambda_0^{(1)}, \lambda_1^{(1)}, \dots, \lambda_{n-1}^{(1)}, \lambda_n^{(1)} \in R$ and $\lambda_n^{(1)} = n!$. The product matrix $n(zI - A)(zI - A)^*$ can be written as

$$n(zI - A)(zI - A)^* = p_{A,1}(z)I + C_0 + C_1z + \cdots + C_nz^n,$$

where the matrices $C_i \in M_n(R)$ are uniquely determined by A , $\text{tr}(C_i) = 0$ and each entry of C_i is in $[R, R]$, i.e. $C_i \in M_n([R, R])$ for all $0 \leq i \leq n$. The right

$$(\lambda_0^{(1)}I + C_0) + A(\lambda_1^{(1)}I + C_1) + \cdots + A^{n-1}(\lambda_{n-1}^{(1)}I + C_{n-1}) + A^n(n!I + C_n) = 0$$

and a similar left

$$(\mu_0^{(1)}I + D_0) + (\mu_1^{(1)}I + D_1)A + \cdots + (\mu_{n-1}^{(1)}I + D_{n-1})A^{n-1} + (n!I + D_n)A^n = 0$$

Cayley-Hamilton identity with right and left matrix coefficients hold for A .

Theorem 2.7. (see [8]) *If the ring R satisfies the polynomial identity*

$$[[[\dots[x_1, x_2], x_3], \dots], x_k], x_{k+1}] = 0$$

(R is Lie nilpotent of index k), then the k -th right characteristic polynomial $p_{A,k}(z) \in R[z]$ of a matrix $A \in M_n(R)$ is of the form

$$p_{A,k}(x) = \lambda_0^{(k)} + \lambda_1^{(k)}x + \cdots + \lambda_{n^k-1}^{(k)}x^{n^k-1} + \lambda_{n^k}^{(k)}x^{n^k},$$

with $\lambda_0^{(k)}, \lambda_1^{(k)}, \dots, \lambda_{n^k-1}^{(k)}, \lambda_{n^k}^{(k)} \in R$ and $\lambda_{n^k}^{(k)} = n\{(n-1)!\}^{1+n+n^2+\dots+n^{k-1}}$. The right

$$(A)p_{A,k} = I\lambda_0^{(k)} + A\lambda_1^{(k)} + \cdots + A^{n^k-1}\lambda_{n^k-1}^{(k)} + A^{n^k}\lambda_{n^k}^{(k)} = 0$$

and a similar left

$$q_{A,k}(A) = \mu_0^{(k)}I + \mu_1^{(k)}A + \cdots + \mu_{n^k-1}^{(k)}A^{n^k-1} + \mu_{n^k}^{(k)}A^{n^k} = 0$$

Cayley-Hamilton identity with right and left scalar coefficients hold for A .

The Cayley-Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $M_n(K)$ over a field K (see [2],[3],[6],[7]). In case of $\text{char}(K) = 0$, Kemer's pioneering work (see [4]) on the T-ideals of associative algebras revealed the importance of the identities satisfied by $M_n(E)$ and $M_{n,t}(E)$, where

$$E = K \langle v_1, v_2, \dots, v_i, \dots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle = E_0 \oplus E_1$$

is the naturally \mathbb{Z}_2 -graded Grassmann (exterior) algebra generated by the infinite sequence of anticommutative indeterminates $(v_i)_{i \geq 1}$ (E is Lie nilpotent of index 2). Let $K \langle x_1, x_2, \dots, x_i, \dots \rangle$ denote the polynomial K -algebra generated by the infinite sequence $x_1, x_2, \dots, x_i, \dots$ of non-commuting indeterminates. The prime T-ideals of this (free associative K -)algebra are exactly the T-ideals of the identities satisfied by $M_n(K)$ for $n \geq 1$. The T-prime T-ideals are the prime T-ideals plus the T-ideals of the identities of $M_n(E)$ for $n \geq 1$ and of $M_{n,t}(E)$ for $n-1 \geq t \geq 1$. Another remarkable result is that for a sufficiently large $n \geq 1$, any T-ideal contains the T-ideal of the identities satisfied by $M_n(E)$. Accordingly, the importance of

matrices (and supermatrices) over certain non-commutative rings is an evidence in the theory of PI-rings, nevertheless this fact has been obvious for a long time in other branches of algebra (e.g. in the structure theory of semisimple rings). Thus the algebras $M_n(E)$ and $M_{n,t}(E)$ served as the main motivation for the development of the above presented results. We also note that for 2×2 matrices over a so called Lie-solvable base ring a kind of Cayley-Hamilton trace identity was exhibited in [5].

3. THE SYMMETRIC DETERMINANT AND THE SYMMETRIC CHARACTERISTIC POLYNOMIAL

If the base ring R is commutative, then $\text{tr}(AB) = \text{tr}(BA)$ for all $A, B \in M_n(R)$. In spite of the fact that this well known trace identity is no longer valid for matrices over a non-commutative ring, the first left and first right determinants of A coincide (it was not recognized in [10]).

Theorem 3.1. *The traces of the product matrices A^*A and AA^* are both equal to the symmetric determinant of A :*

$$\text{rdet}_{(1)}(A) = \text{tr}(AA^*) = \text{sdet}(A) = \text{tr}(A^*A) = \text{ldet}_{(1)}(A).$$

Proof. We prove that $\text{tr}(AA^*) = \text{sdet}(A)$, the proof of $\text{sdet}(A) = \text{tr}(A^*A)$ is similar. The trace of a matrix is the sum of the diagonal entries, hence

$$\begin{aligned} \text{tr}(A^*A) &= \sum_{1 \leq r, s \leq n} a_{r,s}^* a_{s,r} \\ &= \sum_{(\alpha, \beta, s) \in \Delta_n} \text{sgn}(\alpha) \text{sgn}(\beta) a_{\alpha(1), \beta(1)} \cdots a_{\alpha(s-1), \beta(s-1)} a_{\alpha(s+1), \beta(s+1)} \cdots a_{\alpha(n), \beta(n)} a_{\alpha(s), \beta(s)} \\ &= \sum_{\alpha', \beta' \in S_n} \text{sgn}(\alpha') \text{sgn}(\beta') a_{\alpha'(1), \beta'(1)} \cdots a_{\alpha'(t), \beta'(t)} \cdots a_{\alpha'(n), \beta'(n)} = \text{sdet}(A), \end{aligned}$$

where $\Delta_n = \{(\alpha, \beta, s) \mid \alpha, \beta \in S_n, 1 \leq s \leq n, \alpha(s) = s\}$ and the map $(\alpha, \beta, s) \mapsto (\alpha', \beta')$ with

$$\alpha' = \begin{pmatrix} 1 & \cdots & s-1 & s & \cdots & n-1 & n \\ \alpha(1) & \cdots & \alpha(s-1) & \alpha(s+1) & \cdots & \alpha(n) & s \end{pmatrix}$$

and

$$\beta' = \begin{pmatrix} 1 & \cdots & s-1 & s & \cdots & n-1 & n \\ \beta(1) & \cdots & \beta(s-1) & \beta(s+1) & \cdots & \beta(n) & \beta(s) \end{pmatrix}$$

is a $\Delta_n \rightarrow S_n \times S_n$ bijection. Since

$$\text{sgn}(\alpha') = (-1)^{n-s} \text{sgn}(\alpha), \quad \text{sgn}(\beta') = (-1)^{n-s} \text{sgn}(\beta)$$

and

$$\begin{aligned} &a_{\alpha(1), \beta(1)} \cdots a_{\alpha(s-1), \beta(s-1)} a_{\alpha(s+1), \beta(s+1)} \cdots a_{\alpha(n), \beta(n)} a_{\alpha(s), \beta(s)} \\ &= a_{\alpha'(1), \beta'(1)} \cdots a_{\alpha'(t), \beta'(t)} \cdots a_{\alpha'(n), \beta'(n)}, \end{aligned}$$

the proof is complete. \square

Corollary 3.2. *The first right and left characteristic polynomials of a matrix $A \in M_n(R)$ coincide: $p_{A,1}(z) = q_{A,1}(z)$. Thus we have $\lambda_i^{(1)} = \mu_i^{(1)}$ for all $0 \leq i \leq n$ in the corresponding Cayley-Hamilton identities (see Theorem 2.6).*

In view of Theorem 3.1 and Corollary 3.2, for the above determinants and characteristic polynomials, it is reasonable to use the terminology "symmetric" instead of "first right" and "first left".

The following observation for 2×2 matrices over the Grassmann algebra is due to Domokos (see [1]).

Proposition 3.3. *If $A = [a_{i,j}]$ is in $M_2(R)$, then*

$$\text{rdet}_{(2)}(A) - \text{ldet}_{(2)}(A) = \mathcal{S}_4(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}),$$

where $\mathcal{S}_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} \text{sgn}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$ is the standard polynomial of degree four.

Proof. Using

$$A^* = \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}$$

and the products

$$AA^* = \begin{bmatrix} a_{1,1}a_{2,2} - a_{1,2}a_{2,1} & -a_{1,1}a_{1,2} + a_{1,2}a_{1,1} \\ a_{2,1}a_{2,2} - a_{2,2}a_{2,1} & -a_{2,1}a_{1,2} + a_{2,2}a_{1,1} \end{bmatrix},$$

$$A^*A = \begin{bmatrix} a_{2,2}a_{1,1} - a_{1,2}a_{2,1} & a_{2,2}a_{1,2} - a_{1,2}a_{2,2} \\ -a_{2,1}a_{1,1} + a_{1,1}a_{2,1} & -a_{2,1}a_{1,2} + a_{1,1}a_{2,2} \end{bmatrix},$$

a direct computation shows that

$$\begin{aligned} \text{rdet}_{(2)}(A) - \text{ldet}_{(2)}(A) &= \text{rdet}_{(1)}(AA^*) - \text{ldet}_{(1)}(A^*A) = \text{sdet}(AA^*) - \text{sdet}(A^*A) \\ &= (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})(-a_{2,1}a_{1,2} + a_{2,2}a_{1,1}) + (-a_{2,1}a_{1,2} + a_{2,2}a_{1,1})(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \\ &\quad - (-a_{1,1}a_{1,2} + a_{1,2}a_{1,1})(a_{2,1}a_{2,2} - a_{2,2}a_{2,1}) - (a_{2,1}a_{2,2} - a_{2,2}a_{2,1})(-a_{1,1}a_{1,2} + a_{1,2}a_{1,1}) \\ &\quad - (a_{2,2}a_{1,1} - a_{1,2}a_{2,1})(-a_{2,1}a_{1,2} + a_{1,1}a_{2,2}) - (-a_{2,1}a_{1,2} + a_{1,1}a_{2,2})(a_{2,2}a_{1,1} - a_{1,2}a_{2,1}) \\ &\quad + (a_{2,2}a_{1,2} - a_{1,2}a_{2,2})(-a_{2,1}a_{1,1} + a_{1,1}a_{2,1}) + (-a_{2,1}a_{1,1} + a_{1,1}a_{2,1})(a_{2,2}a_{1,2} - a_{1,2}a_{2,2}) \\ &= \mathcal{S}_4(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}). \quad \square \end{aligned}$$

Corollary 3.4. *If $A = [a_{i,j}]$ is in $M_2(R)$, then*

$$\begin{aligned} p_{A,2}(z) - q_{A,2}(z) &= \text{rdet}_{(2)}(zI - A) - \text{ldet}_{(2)}(zI - A) \\ &= \mathcal{S}_4(z - a_{1,1}, -a_{1,2}, -a_{2,1}, z - a_{2,2}) = \mathcal{S}_4(-a_{1,1}, -a_{1,2}, -a_{2,1}, -a_{2,2}) \\ &= \mathcal{S}_4(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) \end{aligned}$$

is a constant polynomial in $R[z]$.

4. THE SYMMETRIC NEWTON FORMULAE FOR 2×2 AND 3×3 MATRICES

If our base ring R is commutative, then the well known Newton trace formulae for 2×2 and 3×3 matrices are the following

$$\begin{aligned} 2 \det(A) &= \text{tr}^2(A) - \text{tr}(A^2), \\ 6 \det(A) &= \text{tr}^3(A) - 3\text{tr}(A)\text{tr}(A^2) + 2\text{tr}(A^3). \end{aligned}$$

Proposition 4.1. *If R is an arbitrary ring and $A \in M_2(R)$, then the symmetric analogue*

$$\text{sdet}(A) = \text{tr}^2(A) - \text{tr}(A^2)$$

of the classical 2×2 Newton formula holds. Notice that $\text{sdet}(A) = 2 \det(A)$ in case of a commutative R .

Proof. Using

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix},$$

we obtain that

$$\text{tr}^2(A) - \text{tr}(A^2) = (a + d)^2 - (a^2 + bc + cb + d^2) = ad + da - bc - cb = \text{sdet}(A). \quad \square$$

Theorem 4.2. *If R is an arbitrary ring and $A \in M_3(R)$, then the following symmetric analogue of the classical 3×3 Newton formula holds:*

$$\text{sdet}(A) = \text{tr}^3(A) - \text{tr}(A) \cdot \text{tr}(A^2) - \text{tr}(A \cdot \text{tr}(A) \cdot A) - \text{tr}(A^2) \cdot \text{tr}(A) + \text{tr}(A^3) + \text{tr}((A^\top)^3).$$

Notice that

$$\text{sdet}(A) = 6 \det(A), \quad \text{tr}(A) \text{tr}(A^2) = \text{tr}(A \cdot \text{tr}(A) \cdot A) = \text{tr}(A^2) \cdot \text{tr}(A), \quad \text{tr}((A^\top)^3) = \text{tr}(A^3)$$

in case of a commutative R .

Proof. Using

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & p \end{bmatrix}, \quad A^\top = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & p \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a^2 + bd + cg & ab + be + ch & ac + bf + cp \\ da + ed + fg & db + e^2 + fh & dc + ef + fp \\ ga + hd + pg & gb + he + ph & gc + hf + p^2 \end{bmatrix}$$

and

$$A^3 = \begin{bmatrix} a^2 + bd + cg & ab + be + ch & ac + bf + cp \\ da + ed + fg & db + e^2 + fh & dc + ef + fp \\ ga + hd + pg & gb + he + ph & gc + hf + p^2 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & p \end{bmatrix},$$

we obtain that

$$\text{tr}(A^2) = a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2$$

and

$$\begin{aligned} \text{tr}(A^3) &= (a^2 + bd + cg)a + (ab + be + ch)d + (ac + bf + cp)g \\ &\quad + (da + ed + fg)b + (db + e^2 + fh)e + (dc + ef + fp)h \\ &\quad + (ga + hd + pg)c + (gb + he + ph)f + (gc + hf + p^2)p. \end{aligned}$$

Performing the transpositions $b \leftrightarrow d$, $c \leftrightarrow g$, $f \leftrightarrow h$ in the above expression for $\text{tr}(A^3)$, we deduce that

$$\begin{aligned} \text{tr}((A^\top)^3) &= (a^2 + db + gc)a + (ad + de + gf)b + (ag + dh + gp)c \\ &\quad + (ba + eb + hc)d + (bd + e^2 + hf)e + (bg + eh + hp)f \\ &\quad + (ca + fb + pc)g + (cd + fe + pf)h + (cg + fh + p^2)p. \end{aligned}$$

Hence,

$$\begin{aligned} \text{tr}^3(A) - \text{tr}(A) \cdot \text{tr}(A^2) - \text{tr}(A \cdot \text{tr}(A) \cdot A) - \text{tr}(A^2) \cdot \text{tr}(A) + \text{tr}(A^3) + \text{tr}((A^\top)^3) \\ = (a + e + p)(a + e + p)(a + e + p) \\ - (a + e + p)(a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2) \\ - a(a + e + p)a - b(a + e + p)d - c(a + e + p)g \\ - d(a + e + p)b - e(a + e + p)e - f(a + e + p)h \\ - g(a + e + p)c - h(a + e + p)f - p(a + e + p)g \end{aligned}$$

$$\begin{aligned}
& -(a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2)(a + e + p) \\
& + (a^2 + bd + cg)a + (ab + be + ch)d + (ac + bf + cp)g \\
& + (da + ed + fg)b + (db + e^2 + fh)e + (dc + ef + fp)h \\
& + (ga + hd + pg)c + (gb + he + ph)f + (gc + hf + p^2)p \\
& + (a^2 + db + gc)a + (ad + de + gf)b + (ag + dh + gp)c \\
& + (ba + eb + hc)d + (bd + e^2 + hf)e + (bg + eh + hp)f \\
& + (ca + fb + pc)g + (cd + fe + pf)h + (cg + fh + p^2)p \\
& = a^3 + a^2e + a^2p + aea + ae^2 + aep + apa + ape + ap^2 \\
& + ea^2 + eae + eap + e^2a + e^3 + e^2p + epa + epe + ep^2 \\
& + pa^2 + pa e + pap + pea + pe^2 + pep + p^2a + p^2e + p^3 \\
& - a(a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2) \\
& - e(a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2) \\
& - p(a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2) \\
& - a(a + e + p)a - b(a + e + p)d - c(a + e + p)g \\
& - d(a + e + p)b - e(a + e + p)e - f(a + e + p)h \\
& - g(a + e + p)c - h(a + e + p)f - p(a + e + p)p \\
& - (a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2)a \\
& - (a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2)e \\
& - (a^2 + bd + cg + db + e^2 + fh + gc + hf + p^2)p \\
& + (a^2 + bd + cg)a + (ab + be + ch)d + (ac + bf + cp)g \\
& + (da + ed + fg)b + (db + e^2 + fh)e + (dc + ef + fp)h \\
& + (ga + hd + pg)c + (gb + he + ph)f + (gc + hf + p^2)p \\
& + (a^2 + db + gc)a + (ad + de + gf)b + (ag + dh + gp)c \\
& + (ba + eb + hc)d + (bd + e^2 + hf)e + (bg + eh + hp)f \\
& + (ca + fb + pc)g + (cd + fe + pf)h + (cg + fh + p^2)p \\
& = aep + ape + eap + epa + pa e + pea \\
& + bfg + bgf + fbg + fgb + gbf + gfb \\
& + cdh + chd + dch + dhc + hcd + hdc \\
& - ceg - cge - ecg - egc - gce - gec \\
& - afh - ahf - fah - fha - haf - hfa \\
& - bdp - bpd - dbp - dpb - pbd - pdb \\
& = \text{sdet}(A). \quad \square
\end{aligned}$$

4.3. Remark. For a non-commutative ring R , the identity $\text{tr}((A^\top)^2) = \text{tr}(A^2)$ holds for any $A \in M_n(R)$, but $\text{tr}((A^\top)^3) = \text{tr}(A^3)$ is not valid even in the 2×2 case.

4.4. Theorem. If A is a 3×3 matrix over an arbitrary ring R , then the symmetric characteristic polynomial of A in $R[z]$ is

$$p_{A,1}(z) = q_{A,1}(z) = \text{sdet}(zI - A) = 6z^3 - 6\text{tr}(A)z^2 + 3(\text{tr}^2(A) - \text{tr}(A^2))z - \text{sdet}(A).$$

Proof.

$$\text{tr}(zI - A) = \text{tr}(zI) - \text{tr}(A) = 3z - \text{tr}(A),$$

$$\begin{aligned}
\text{tr}^3(zI - A) &= (3z - \text{tr}(A))^3 = 27z^3 - 27\text{tr}(A)z^2 + 9\text{tr}^2(A)z - \text{tr}^3(A), \\
(zI - A)^2 &= z^2I - zA - Az + A^2, \quad \text{tr}((zI - A)^2) = 3z^2 - 2\text{tr}(A)z + \text{tr}(A^2), \\
\text{tr}(zI - A) \cdot \text{tr}((zI - A)^2) &= (3z - \text{tr}(A)) \cdot (3z^2 - 2\text{tr}(A)z + \text{tr}(A^2)) \\
&= 9z^3 - 9\text{tr}(A)z^2 + 2\text{tr}^2(A)z + 3\text{tr}(A^2)z - \text{tr}(A)\text{tr}(A^2), \\
\text{tr}((zI - A)^2) \cdot \text{tr}(zI - A) &= (3z^2 - 2\text{tr}(A)z + \text{tr}(A^2)) \cdot (3z - \text{tr}(A)) \\
&= 9z^3 - 9\text{tr}(A)z^2 + 2\text{tr}^2(A)z + 3\text{tr}(A^2)z - \text{tr}(A^2)\text{tr}(A), \\
(zI - A) \cdot \text{tr}(zI - A) \cdot (zI - A) &= (zI - A) \cdot (3z - \text{tr}(A)) \cdot (zI - A) \\
&= 3Iz^3 - 3Az^2 - \text{tr}(A)Iz^2 - 3Az^2 + \text{tr}(A)Az + 3A^2z + \text{Atr}(A)z - \text{Atr}(A)A, \\
\text{tr}((zI - A) \cdot \text{tr}(zI - A) \cdot (zI - A)) &= 9z^3 - 9\text{tr}(A)z^2 + 2\text{tr}^2(A)z + 3\text{tr}(A^2)z - \text{tr}(A \cdot \text{tr}(A) \cdot A), \\
(zI - A)^3 &= z^3I - z^2A - zAz - Az^2 + A^2z + AzA + zA^2 - A^3, \\
\text{tr}((zI - A)^3) &= 3z^3 - 3\text{tr}(A)z^2 + 3\text{tr}(A^2)z - \text{tr}(A^3), \\
\text{tr}((zI - A)^\top)^3 &= \text{tr}((zI - A^\top)^3) = 3z^3 - 3\text{tr}(A^\top)z^2 + 3\text{tr}((A^\top)^2)z - \text{tr}((A^\top)^3) \\
&= 3z^3 - 3\text{tr}(A)z^2 + 3\text{tr}(A^2)z - \text{tr}((A^\top)^3).
\end{aligned}$$

The application of Theorem 4.2 gives that

$$\begin{aligned}
\text{sdet}(zI - A) &= \text{tr}^3(zI - A) \\
&- \text{tr}(zI - A) \cdot \text{tr}((zI - A)^2) - \text{tr}((zI - A) \cdot \text{tr}(zI - A) \cdot (zI - A)) - \text{tr}((zI - A)^2) \cdot \text{tr}(zI - A) \\
&+ \text{tr}((zI - A)^3) + \text{tr}(((zI - A)^\top)^3) = [27z^3 - 27\text{tr}(A)z^2 + 9\text{tr}^2(A)z - \text{tr}^3(A)] \\
&- [9z^3 - 9\text{tr}(A)z^2 + 2\text{tr}^2(A)z + 3\text{tr}(A^2)z - \text{tr}(A)\text{tr}(A^2)] \\
&- [9z^3 - 9\text{tr}(A)z^2 + 2\text{tr}^2(A)z + 3\text{tr}(A^2)z - \text{tr}(A \cdot \text{tr}(A) \cdot A)] \\
&- [9z^3 - 9\text{tr}(A)z^2 + 2\text{tr}^2(A)z + 3\text{tr}(A^2)z - \text{tr}(A^2)\text{tr}(A)] \\
&+ [3z^3 - 3\text{tr}(A)z^2 + 3\text{tr}(A^2)z - \text{tr}(A^3)] + [3z^3 - 3\text{tr}(A)z^2 + 3\text{tr}(A^2)z - \text{tr}((A^\top)^3)] \\
&= 6z^3 - 6\text{tr}(A)z^2 + 3(\text{tr}^2(A) - \text{tr}(A^2))z \\
&- \text{tr}^3(A) + \text{tr}(A)\text{tr}(A^2) + \text{tr}(A \cdot \text{tr}(A) \cdot A) + \text{tr}(A^2)\text{tr}(A) - \text{tr}(A^3) - \text{tr}((A^\top)^3) \\
&= 6z^3 - 6\text{tr}(A)z^2 + 3(\text{tr}^2(A) - \text{tr}(A^2))z - \text{sdet}(A). \quad \square
\end{aligned}$$

4.5. Corollary. *If $A \in M_3(R)$, then Theorem 2.6 gives the existence of 3×3 matrices C_i, D_i ($0 \leq i \leq 3$) with entries in $[R, R]$ such that*

$$(-\text{sdet}(A)I + C_0) + (3(\text{tr}^2(A) - \text{tr}(A^2))I + C_1)A + (-6\text{tr}(A)I + C_2)A^2 + (6I + C_3)A^3 = 0$$

and

$$(-\text{sdet}(A)I + D_0) + A(3(\text{tr}^2(A) - \text{tr}(A^2))I + D_1) + A^2(-6\text{tr}(A)I + D_2) + A^3(6I + D_3) = 0.$$

4.6. Corollary. *If $\frac{1}{6} \in R$ and $A \in M_3(R)$ such that*

$$\text{tr}(A) = \text{tr}(A^2) = \text{tr}(A^3) = \text{tr}((A^\top)^3) = 0,$$

then $\text{sdet}(A) = 0$ and

$$A^3 = C_0 + C_1A + C_2A^2 + C_3A^3 = D_0 + AD_1 + A^2D_2 + A^3D_3$$

for some 3×3 matrices C_i, D_i ($0 \leq i \leq 3$) with entries in $[R, R]$. Thus $A^3 \in M_3(T)$, where $T = R[R, R] \cap [R, R]R$ is the intersection of the left and right ideals $R[R, R]$ and $[R, R]R$ of R .

We close Section 4 by the following.

4.7. Problem. *If R is a commutative ring, then the Newton formula for a 4×4 matrix $A \in M_4(R)$ is*

$$24 \det(A) = \operatorname{tr}^4(A) - 6 \operatorname{tr}^2(A) \operatorname{tr}(A^2) + 3 \operatorname{tr}^2(A^2) + 8 \operatorname{tr}(A) \operatorname{tr}(A^3) - 6 \operatorname{tr}(A^4).$$

Find the symmetric analogue of the above formula for $\operatorname{sdet}(A)$ over an arbitrary ring R .

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF MISKOLC, MISKOLC, HUNGARY 3515
E-mail address: jeno.szigeti@uni-miskolc.hu

DEPARTMENT OF MATHEMATICAL SCIENCES, STELLENBOSCH UNIVERSITY, P/BAG X1, MATIELAND
 7602, STELLENBOSCH, SOUTH AFRICA
E-mail address: LvW@sun.ac.za